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Topological Defects in Superfluid Helium

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Some aspects of classical field-theoretic phenomenology of superfluid helium are presented.

1. INTRODUCTION

The aim of this paper is to present a description of superfluid helium in terms of classical field theory (CFT). We apply well-known facts from CFT to shed new light on some familiar phenomena. Our approach is formally identical to that used in the theory of superconductors. However, from our point of view some aspects of this interpretation are not sufficiently well stressed in the literature related to the theory of superfluids.

2. RELATIVISTIC APPROACH TO A DYNAMICS OF SUPERFLUID HELIUM

It is a well-established practice to describe many features of condensed matter in terms of an order parameter. For superfluid helium this role is played by a scalar complex-valued field, similar in many aspects to the ordinary wave function considered in quantum mechanics. The theory of superfluid helium is usually treated as unrelativistic. Our approach, inspired by Anandan and Morrison (1981; Putterman, 1974) is consequently relativistic.

Let us start the discussion from the construction of the Lagrangian. Let $\varphi(x)$ denote an order parameter for superfluid helium. The simplest form of a Lorentz-invariant dynamical term for a complex-valued scalar field is

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$g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}$, where $g_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$. Conventionally, $g_{\mu\nu}$ is introduced by the relation $ds^2=g_{\mu\nu}\,dx^\mu\,dx^\nu=c^2\,dt^2-dx^2-dy^2-dz^2$, where c is the velocity of light. In our approach c is no longer the velocity of light, but rather the velocity of the first sound in superfluid helium. The reasons for making this assumption will become clear in the further discussion of the field equations. A full description requires that we add a potential term to the Lagrangian. A simple model of superfluid helium treats it as a weak interacting Bose gas, which can be described in terms of some $\lambda\varphi^4$ field theory. Then the potential term takes the form

$$V(\varphi\varphi^*)=\lambda\,(\varphi_0^2-\varphi\varphi^*)^2, \qquad \lambda,\ \varphi_0=\mathrm{const}, \quad \lambda>0$$

The Lagrangian of "superfluid field theory" takes the form

$$\mathscr{L}=\partial_\mu\varphi\,\partial^\mu\varphi^*-V(\varphi\varphi^*)$$

This Lagrangian was constructed to be Lorentz invariant. Another symmetry is its global $U(1)$ symmetry: when φ is transformed by $\varphi\to e^{i\alpha}\varphi$, $\alpha\in\mathbb{R}$, $\alpha=\mathrm{const}$, the Lagrangian is unchanged.

Further discussion of the field equations and their properties is based on Huang (1982).

By a procedure of variation of φ one obtains the field equation

$$\partial_\mu\,\partial^\mu\varphi=-\frac{\partial V}{\partial\varphi^*}=-2\lambda\,(\varphi_0^2-|\varphi|^2)\varphi \tag{1}$$

which is a nonlinear Klein–Gordon equation.

3. THE ROLE OF A GLOBAL SYMMETRY BREAKING

It is interesting to investigate solutions of (1) possessing the lowest energy. The Hamiltonian of our field is as follows:

$$H=\int d^3x\left[\frac{1}{c^2}\left|\frac{\partial\varphi}{\partial t}\right|^2+|\nabla\varphi|^2+V(\varphi\varphi^*)\right]$$

The solution minimizing this Hamiltonian takes the form $\varphi=\varphi_0\,e^{i\alpha_0}$, where $\alpha_0\in\mathbb{R}$ is arbitrary. It is easy to check that this solution is not $U(1)$ invariant. This phenomenon is called in the field theory a spontaneous symmetry breaking.

From the Goldstone theorem it is known that the spontaneous symmetry breaking of any global continuous symmetry is accompanied by a generation of a new massless and spinless particle.

For superfluid helium this theorem is fulfilled. To see that, let us consider slightly excited modes of the form

$$\varphi(x) = [\varphi_0 + \eta(x)]\, e^{i\alpha(x)}$$

where $\eta(x)$ and $\alpha(x)$ are new real-valued fields, which are assumed to be small. The Lagrangian for these modes, limited to terms of second order in η, α, takes the form

$$\mathscr{L} = \partial_\mu \eta\, \partial^\mu \eta - 4\lambda\varphi_0^2\eta^2 + \varphi_0^2\, \partial_\mu \alpha\, \partial^\mu \alpha$$

One sees that the field η describes free scalar particles of the mass $m = (2\hbar/c)\lambda^{1/2}\varphi_0$, and the field α describes free scalar particles of zero mass predicted by the Goldstone theorem. We interpret these new particles as phonons, whose dynamics is governed by the wave equation

$$\partial_\mu\, \partial^\mu \alpha = 0$$

Therefore the velocity c defined by the metric tensor

$$c^2\, dt^2 - dx^2 - dy^2 - dz^2$$

must be interpreted as the velocity of sound in the wave equation.

4. FROM PHONON TO ROTON BRANCH OF SUPERFLUID HELIUM

The model described above is useful in the phonon branch of superfluid helium; the roton branch still needs a description. Surprisingly, the description of rotons is possible in similar Lagrangian theory if we extend the global $U(1)$ symmetry to the local one.

In this procedure new fields called "compensating" or "gauge" have to be introduced, to fulfill the required invariance.

Local invariance means invariance with respect to transformations of the form

$$\varphi(x) \to e^{-i\alpha(x)}\varphi(x) \tag{2}$$

Let us start by changing partial derivatives into covariant ones, to obtain the required symmetry:

$$\partial_\mu \varphi \to D_\mu \varphi = \left(\partial_\mu + i\frac{m}{\hbar}\, v_\mu\right)\varphi$$

where m is the mass of a helium atom, and v_μ is the gauge field, which transforms with respect to (2) in the following way:

$$v_\mu \to v_\mu + \frac{\hbar}{m}\,\partial_\mu \alpha$$

The Lagrangian

$$\mathscr{L}(\varphi, \varphi_{,\mu}, v_\mu) = (D_\mu \varphi)(D^\mu \varphi)^* - V(\varphi\varphi^*)$$

has to be completed by dynamical terms of the field v_μ. Since the Lorentz invariance and $U(1)$ local invariance are required, we introduce the term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ into the Lagrangian, where $F_{\mu\nu} = v_{\nu,\mu} - v_{\mu,\nu}$, and obtain a linear equation for v_μ.

Finally, we obtain the Lagrangian of the form

$$\mathscr{L}(\varphi, \varphi_{,\mu}, v_\mu, v_{\mu,\nu}) = (D_\mu \varphi)(D^\mu \varphi)^* - V(\varphi\varphi^*) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where $V(\varphi\varphi^*)$ is of the same form as above.

Formally this Lagrangian is identical to the Lagrangian of the scalar electrodynamics. The field equations are of the following form:

$$\begin{aligned} D_\mu D^\mu \varphi &= -\frac{\partial V}{\partial \varphi^*} = -2\lambda\,(\varphi_0^2 - \varphi\varphi^*)\varphi \\ \partial_\nu F^{\mu\nu} &= j^\mu \end{aligned} \tag{3}$$

where

$$j^\mu = i\frac{m}{\hbar}\,\varphi^* \overleftrightarrow{D^\mu}\varphi = i\frac{m}{\hbar}\,\varphi^* \overleftrightarrow{\partial^\mu}\varphi - 2\frac{m^2}{\hbar^2}\,(\varphi\varphi^*)v^\mu$$

is the conserved current: $\partial_\mu j^\mu = \partial_\mu \partial_\nu F^{\mu\nu} \equiv 0$.

We are looking for solutions to the system (3) possessing the lowest energy. Let $\bar{D}$ and $\bar{\omega}$ be fields defined as follows:

$$\begin{aligned} D^i &= F^{i0} \\ \omega^i &= -\tfrac{1}{2}\varepsilon^{ijk}F^{jk} \end{aligned}$$

The Hamiltonian of our system is of the form

$$H = \int d^3x\,[\tfrac{1}{2}(\bar{\omega}^2 + \bar{D}^2) + |\pi|^2 + |\bar{D}\varphi|^2 + V(\varphi\varphi^*)]$$

where $\pi = (D^0\varphi)^*$ is the momentum conjugated to φ, $\bar{D}\varphi$ denotes the spatial part of $D^\mu\varphi$, and boundary terms were omitted because we assume the considered fields to be of limited range.

From the form of the Hamiltonian it is obvious that the solution of (3) possessing the lowest energy is of the form:

$$v^\mu(x) = 0$$

$$\varphi(x) = \varphi_0\, e^{i\alpha_0}, \qquad \alpha_0 \text{ is arbitrary}$$

This solution is also not invariant with respect to local $U(1)$ transformations.

It is convenient to use a unitary gauge to describe the excited classical modes of this solution. This gauge is introduced in such a way that the field φ becomes real and its phase is compensated by the gauge transformation

$$\varphi(x) = \rho(x), \qquad \rho \text{ is real}$$

In such a gauge, slightly excited modes take the form

$$\rho(x) = \varphi_0 + \eta(x)$$

$$v^\mu(x) = 0 + v^\mu(x)$$

where η and v^μ are considered to be small.

The linearized field equations in terms of the variables defined above are of the form

$$\left(\Box + 2\frac{m^2}{\hbar^2}\varphi_0^2\right) v^\mu = 0 \tag{4}$$

$$(\Box + 4\lambda\varphi_0^2)\eta = 0 \tag{5}$$

Equation (4) describes particles of spin 1 and mass $m_v = 2^{1/2}(m/\hbar)\varphi_0$, and equation (5) describes particles of spin 0 and mass $m_\eta = 2\lambda^{1/2}\varphi_0$. In contrast to spontaneous global symmetry breaking, a spontaneous local symmetry breaking does not generate any new particles.

There is a very interesting physical interpretation of the massiveness of the gauge field v^μ. The field strength $F^{\mu\nu}$ is zero in superfluid helium except in very thin areas of a size of about the Compton wavelength of the gauge field, which is inversely proportional to the mass m_v of the gauge field: $\lambda_c = \hbar/m_v c$. If rotons play the role of carriers of the gauge field ($m_v = 0.16 m_{He}$), and the velocity of sound is equal to $c = 238$ m/sec, the Compton wavelength is $\lambda_c \approx 1$ Å. It is a quantity of the same order as the experimentally measured sizes of vortices in superfluid helium.

5. QUANTUM VORTICES AS TOPOLOGICAL DEFECTS

Classical field theory of superfluid helium is a good tool to describe quantum vortices, as will be seen in further discussion of the field equations. There exist solutions of field equations of the vortex form. These solutions are topologically nontrivial and can be described in terms of homotopy groups.

Let me begin from topological considerations. The possibility of the existence of topological defects of a dimension k ($k=0, 1, 2$) in an arbitrary theory is characterized by the nontriviality of the $(2-k)$th homotopy group of the manifold of ground states, i.e., $\pi_{2-k}(M)\neq 0$ (Kibble, 1976).

For superfluid helium the manifold of ground states is homeomorphic to a circle $M=S^1$, where $S^1=\{\varphi_0\, e^{i\alpha_0}\colon \alpha_0\in\mathbb{R}\}$.

Homotopy groups of rank q for S^1 are the following:

$$\pi_q(S^1)=\begin{cases}0, & q\neq 1\\ \mathbb{Z}, & q=1\end{cases}$$

Hence only one-dimensional defects are possible in superfluid helium. They are identified as quantum vortices.

We are looking for cylindrically symmetric static solutions of field equations of a vortex form. Consider the gauge $v^0(x)=0$ and suitable boundary conditions:

$$\bar{v}(x)\xrightarrow[|x|\to\infty]{}\frac{\hbar}{m}\nabla\alpha(\bar{x}) \qquad \text{(pure gauge)}$$

$$\varphi(x)\xrightarrow[|x|\to\infty]{}\varphi_0\, e^{i\alpha(x)}$$

Consider the boundaries in the form of a cylinder with a large radius R. Let $\alpha(\theta)$, $\varphi(\theta)$, $0\leq\theta\leq 2\pi$, denote $\alpha(\bar{x})$, $\varphi(\bar{x})$ limited to this cylinder. A function $\varphi(\theta)$ must be a continuous one, because it satisfies a differential equation: $\varphi(2\pi)=\varphi(0)\Rightarrow\alpha(2\pi)-\alpha(0)=2\pi n$, $n\in\mathbb{Z}$.

A flux of the gauge field strength $F_{\mu\nu}$ has to be quantized:

$$\Phi=\iint d\sigma^{\mu\nu}\, F_{\mu\nu}=\oint\bar{v}\, d\bar{l}=\frac{h}{m}n, \qquad n\in\mathbb{Z}$$

This is the well-known rule of quantization of the vorticity in superfluid helium.

In the static case, when the gauge $v^0=0$, div $\bar{v}=0$ is taken into consideration, the field equations take the form

$$\left(\nabla^2-2\frac{m^2}{\hbar^2}\varphi^*\varphi\right)\bar{v}=-i\varphi^*\overleftrightarrow{\nabla}\varphi$$

$$\left(\nabla-i\frac{m}{\hbar}\bar{v}\right)^2\varphi=-2\lambda(\varphi_0^2-\varphi^*\varphi)\varphi$$

It is convenient to introduce polar coordinates (r, θ) and to look for solutions of the form

$$\bar{v}(r,\theta)=v(r)\,\bar{e}_\theta$$

$$\varphi(r,\theta)=\rho(r)\,e^{in\theta}, \qquad n\in\mathbb{Z}$$

If one introduces a function $F(r)$ by

$$v(r)=\frac{n}{er}[1-F(r)]$$

then the field equations take the form

$$F''-\frac{F'}{r}-2e^2\rho^2F=0$$

$$\rho''+\frac{\rho'}{r}-\frac{n^2F^2}{r^2}\rho-2\lambda\rho(\rho^2-\varphi_0^2)=0$$

where prime denotes $\partial/\partial r$.

These equations are supplemented by the boundary conditions

$$F(r)\xrightarrow[r\to\infty]{}0$$

$$\rho(r)\xrightarrow[r\to\infty]{}\varphi_0$$

and the condition $F(0)=1$ for $n\neq 0$ must be satisfied to fulfill the condition of quantization of vortex field i.e., $2\pi\int_0^\infty dr\, r\omega(r)=2\pi n/e$, where $\omega(r)$ is defined by

$$\bar{\omega}(r,\theta)=\text{rot}\,\bar{v}=\omega(r)\bar{e}_z$$

One may prove the existence of a solution of this system of equations by variational methods. The asymptotic behavior of such a solution has the following form:

$$\omega \xrightarrow[r\to 0]{} \text{const}$$

$$\rho \xrightarrow[r\to 0]{} 0, \qquad \text{as } r^m \text{ for some } m>0$$

$$\omega \xrightarrow[r\to\infty]{} 0, \qquad \text{as } e^{-r}$$

$$\rho \xrightarrow[r\to\infty]{} \varphi_0$$

From the asymptotic behavior of these solutions one may conclude that they really represent vortexlike objects.

6. VORTICES AS STRINGS

In this part we briefly signal another way to consider one-dimensional objects of vortex form existing in superfluid helium. Namely, there are arguments to treat them as strings, i.e., very thin and long objects possessing a nonzero tension. Nielsen and Olesen (1973) have proved the statement that the action for objects of the vortex form described above is equivalent to the action of a Nambu–Goto string. The idea of a fundamental role of strings in physics is still being intensely investigated. It would be very interesting to use it in the theory of superfluid helium.

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